Math 53: Multivariable Calculus **Sections 201, 204**

Worksheet for 2021-11-08

Let ∇ denote the "operator vector" $\langle \frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z} \rangle$. This is not a vector in the usual sense, since its components are operators rather than scalars, but treating it as one is notationally convenient. For instance, if $f(x,y,z)$ is a scalar function,

$$\nabla f = \langle \frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z} \rangle f = \langle f_x, f_y, f_z \rangle$$

which is the gradient of f. If $\mathbf{F}(x,y,z) = \langle P(x,y,z), Q(x,y,z), R(x,y,z) \rangle$ is a vector field, we can define

$$\nabla \cdot \mathbf{F} = \langle \frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z} \rangle \cdot \langle P, Q, R \rangle = P_x + Q_y + R_z$$

$$\nabla \times \mathbf{F} = \langle \frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z} \rangle \times \langle P, Q, R \rangle = \det \begin{bmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ P & Q & R \end{bmatrix} = \langle R_y - Q_z, P_z - R_x, Q_x - P_y \rangle$$

which are the divergence and curl of $\mathbf{F}$, respectively.

Conceptual questions

Question 1. In the below, f, g denote scalar functions on $\mathbb{R}^3$ while $\mathbf{F}, \mathbf{G}$ denote vector fields on $\mathbb{R}^3$. Which of these expressions do not make sense? For the ones that do make sense, do they output a scalar function or a vector field?

(a) $\operatorname{grad} f$
(b) $\operatorname{curl}(\operatorname{div} \mathbf{F})$
(c) $\operatorname{grad}(\operatorname{div}(\operatorname{curl} \mathbf{F}))$
(d) $\nabla(\nabla \times f)$
(e) $\nabla \cdot (\nabla f)$
(f) $\nabla \cdot (\nabla \times (f\mathbf{G}))$
(g) $\nabla \cdot (\nabla f \times \nabla g)$

Question 2. Some of the expressions in the preceding part make sense but are *always* equal to zero (either the zero function or the zero vector field). Which ones are they?

Hint: Use the vector identity $\nabla \cdot (\mathbf{F} \times \mathbf{G}) = \mathbf{G} \cdot (\nabla \times \mathbf{F}) - \mathbf{F} \cdot (\nabla \times \mathbf{G})$ for (g).

Computations

Problem 1. Let n be a constant, let r denote the scalar function $\sqrt{x^2 + y^2 + z^2}$ (which is different notation from what you are used to; sorry) and let $\mathbf{r}$ denote the vector field $\langle x, y, z \rangle$.

Compute the following:

(a) $\nabla(r^n)$
(b) $\nabla \times (r^{n-1}\mathbf{r})$
(c) $\nabla \cdot (r^{n-1}\mathbf{r})$.

For part (c), what value of n makes the answer equal to zero?

Hint: The most direct way to do this is to just write everything out in terms of x, y, z and differentiate. But if you want to try your hand at using some vector identities instead, these might be helpful:

- It's more convenient to deal with $r^2 = x^2 + y^2 + z^2$ instead of r, since $\nabla(r^2) = \langle 2x, 2y, 2x \rangle = 2\mathbf{r}$ for example.
- If $f(t)$ is a single variable function and $g(x,y,z)$ is a three-variable function, we have the chain rule $\nabla(f(g(x,y,z))) = f'(g(x,y,z))\nabla g(x,y,z)$. What happens if you apply this with $f(t) = t^{n/2}$ and $g = r^2$?
- A product rule: $\nabla \times (f\mathbf{F}) = (\nabla f) \times \mathbf{F} + f(\nabla \times \mathbf{F})$.
- Another product rule: $\nabla \cdot (f\mathbf{F}) = \nabla f \cdot \mathbf{F} + f(\nabla \cdot \mathbf{F})$.